Proc. Indian Acad. Sci. (Math. Sci.) Vol. 114, No. 2, May 2003, pp. 103–106.
Printed in India

A criterion for regular sequences

D P PATIL[1], U STORCH[2] and J STÜCKRAD[3]

[1]Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
[2]Fakultät für Mathematik, Ruhr Universität Bochum, D-44780 Bochum, Germany
[3]Fakultät für Mathematik und Informatik, Universität, Leipzig, D-04109 Leipzig, Germany
E-mail: [1]patil@math.iisc.ernet.in; [2]uwe.storch@ruhr-uni-bochum.de;
[3]stueckrad@mathematik.uni-leipzig.de

MS received 1 January 2004

Abstract. Let R be a commutative noetherian ring and $f_1,\ldots,f_r \in R$. In this article we give (cf. the Theorem in §2) a criterion for $f_1,\ldots,f_r$ to be regular sequence for a finitely generated module over R which strengthens and generalises a result in [2]. As an immediate consequence we deduce that if $\mathrm{V}(g_1,\ldots,g_r) \subseteq \mathrm{V}(f_1,\ldots,f_r)$ in Spec R and if $f_1,\ldots,f_r$ is a regular sequence in R, then $g_1,\ldots,g_r$ is also a regular sequence in R.

Keywords. Regular sequence.

1. Regular sequences

As there is no uniformity about the concept of regular sequence, we first recall the following definitions that we shall use in this note.

DEFINITION 1.

Let R be a commutative noetherian ring and $f_1,\ldots,f_r \in R$. We say that $f_1,\ldots,f_r$ is a *strongly regular sequence* on a R-module M, if for every $i = 1,\ldots,r$ the element f_i is a non-zero divisor for $M/(f_1,\ldots,f_{i-1})M$. The sequence $f_1,\ldots,f_r$ is called a *regular sequence* on a R-module M, if for every $\mathfrak{p} \in \mathrm{Supp}(M/f_1,\ldots,f_r)M)$, the sequence $f_1,\ldots,f_r$ in the local ring $R_{\mathfrak{p}}$ is a strongly regular sequence on the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$.

Note that, in contrast to most of the standard text books, we do not assume the $M \neq (f_1,\ldots,f_r)M$ for a strongly regular sequence $f_1,\ldots,f_r$. For general notations in commutative algebra we also refer to [1].

If the sequence $f_1,\ldots,f_r$ is strongly regular respectively regular on the R-module M, then the same is true for the sequence $f_1\cdot 1_S,\ldots,f_r\cdot 1_S$ on the S-module $S\otimes_R M$, where S is an arbitrary flat noetherian R-algebra.

Note that every sequence is a strongly regular as well as regular sequence on the zero module. Further, it is clear that a strongly regular sequence is a regular sequence but not conversely. For example:

Example. Let $P := k[X,Y,Z]$ be the polynomial ring in three indeterminates over a field k, $\mathfrak{p} := P(X-1)+PZ$, $\mathfrak{q} := PY$ and let $R := P/\mathfrak{p}\cap\mathfrak{q} = P/PY(X-1)+PYZ$. Then Z,X is a regular sequence on the P-module R but not a strongly regular sequence.

The difference between regular and strongly regular sequences is well-illustrated in the following statement given in Chapter II, 6.1 of [4].

PROPOSITION.

Let M be a finitely generated module over a noetherian ring R and let $f_1, \ldots, f_r \in R$. Then the following conditions are equivalent:

(i) $f_1, \ldots, f_r$ *is a strongly regular sequence on M.*
(ii) *For every $s = 1, \ldots, r$ the sequence $f_1, \ldots, f_s$ is a regular sequence on M.*

It can be easily seen that (see the proof of Proposition 3, Chapter IV, A, §1 of [5]) a sequence $f_1, \ldots, f_r$ in a commutative noetherian ring R is a regular sequence for a finitely generated R-module M if and only if the Koszul complex $K_\bullet(f_1, \ldots, f_r; M)$ gives a resolution of $M/(f_1, \ldots, f_r)M$. In particular, if $f_1, \ldots, f_r$ is a regular sequence on M, then for every permutation $\sigma \in \mathfrak{S}_r$ the sequence $f_{\sigma 1}, \ldots, f_{\sigma r}$ is also regular for M. Further, the above proposition implies that the sequence $f_{\sigma 1}, \ldots, f_{\sigma r}$ is strongly regular on M for every $\sigma \in \mathfrak{S}_r$ if and only if all subsequences of $f_1, \ldots, f_r$ are regular on M. For the sake of completeness let us recall Definition 2.

DEFINITION 2.

Let $(R, \mathfrak{m}_R)$ be a noetherian local ring and let M be a non-zero R-module. Then the length of a maximal regular sequence on M in the maximal ideal $\mathfrak{m}_R$ is called the depth of M over R and is denoted by $\mathrm{depth}_R(M)$.

If M is finitely generated then depth can be (cf. [5], Proposition and Definition 3, Chapter IV, A, §2) characterized by

$$(\ddagger) \qquad \mathrm{depth}_R(M) = \min\{i \in \mathbb{N} \mid \mathrm{Ext}^i_R(R/\mathfrak{m}_R, M) \neq 0\}.$$

A finitely generated R-module is called a *Cohen–Macaulay module* if $\dim_R(M) = \mathrm{depth}_R(M)$.

2. Theorem

The following theorem is the main result of this note.

Theorem. *Let R be a commutative noetherian ring, $f_1, \ldots, f_r \in R$ and let M be a finitely generated R-module. Then the following statements are equivalent*:

(i) $f_1, \ldots, f_r$ *is a regular sequence on M.*
(ii) $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq r$ *for every* $\mathfrak{p} \in \mathrm{Supp}(M/(f_1, \ldots, f_r)M)$.
(iii) $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq r$ *for every* $\mathfrak{p} \in \mathrm{Ass}(M/(f_1, \ldots, f_r)M)$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial.

(ii) $\Rightarrow$ (i): We may assume that R is local and $f_1, \ldots, f_r \in \mathfrak{m}_R$. Let $\mathfrak{p} \in \mathrm{Ass}(M)$ and let $\mathfrak{q}$ be a minimal prime ideal in $\mathrm{V}(\mathfrak{p} + Rf_1 + \cdots + Rf_r)$. Then $\mathfrak{q} \in \mathrm{Supp}(M/(f_1, \ldots, f_r)M) = \mathrm{Supp}(M) \cap \mathrm{V}(f_1, \ldots, f_r)$ and so $\mathrm{depth}_{R_\mathfrak{q}} M_\mathfrak{q} \geq r$ by (ii). Since $\mathfrak{p} \in \mathrm{Ass}(M)$, we have

$\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \neq 0$ and so $\mathrm{Ext}^h_{R_\mathfrak{q}}(k(\mathfrak{q}), M_\mathfrak{q}) \neq 0$ by Chapter 6, §18, Lemma 4 of [3], where $h := ht_{R/\mathfrak{p}}(\mathfrak{q}/\mathfrak{p})$. Therefore $r \leq \mathrm{depth}_{R_\mathfrak{q}} M_\mathfrak{q} \leq h$ (see (‡) in §1). But then $f_1 \notin \mathfrak{p}$, since otherwise $h \leq r-1$ by the (generalised) Krull's theorem (see [5], Corollary 4, Chapter III, B, §2). This proves that f_1 is a non-zero divisor for M. Now, induction on r completes the proof.

The implication (iii) $\Rightarrow$ (i) is proved in the lemma which is given below. (In the proof of the lemma we use the implication (ii) $\Rightarrow$ (i).)

COROLLARY 1. ([2], Corollary 1)

Let R be a commutative noetherian ring, $f_1, \ldots, f_r \in R$ and let M be a finitely generated R-module. Then $f_1, \ldots, f_r$ is a regular sequence on M if and only if $f_1, \ldots, f_r$ is a regular sequence on $M_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}(M/f_1, \ldots, f_r)M)$.

COROLLARY 2.

Let R be a commutative noetherian ring and let $f_1, \ldots, f_r, g_1, \ldots, g_r \in R$. Let M be a finitely generated R-module such that $\mathrm{Supp}(M/(g_1, \ldots, g_r)M) \subseteq \mathrm{Supp}(M/(f_1, \ldots, f_r)M)$. Suppose that $f_1, \ldots, f_r$ is a regular sequence on M. Then $g_1, \ldots, g_r$ is also a regular sequence on M. In particular, if $\mathrm{V}(g_1, \ldots, g_r) \subseteq \mathrm{V}(f_1, \ldots, f_r)$ and if $f_1, \ldots, f_r$ is a regular sequence in R, then $g_1, \ldots, g_r$ is also a regular sequence in R.

From the above equivalence we can also deduce the following well-known fact:

COROLLARY 3. (cf. [5], Theorem 2, Chapter IV, B, §2)

If M is a finitely generated Cohen–Macaulay module over a noetherian local ring R, then every system of parameters of M is a regular sequence on M. In particular, in a Cohen–Macaulay local ring every system of parameters is a regular sequence.

Finally, we give a proof of the lemma which we have already used for the proof of the implication (iii) $\Rightarrow$ (i) of the theorem.

Lemma. Let R be a commutative noetherian ring, $f_1, \ldots, f_r \in R$ and let M be a finitely generated R-module. Suppose that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq r$ for every $\mathfrak{p} \in \mathrm{Ass}(M/(f_1, \ldots, f_r)M)$. Then $f_1, \ldots, f_r$ is a regular sequence on M.

Proof. We shall prove by induction on r the following implication:

$(*)_r$: If $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq r$ for every $\mathfrak{p} \in \mathrm{Ass}(M/(f_1, \ldots, f_r)M)$, then $f_1, \ldots, f_r$ is a regular sequence on M.

Proof of $(*)_1$. Put $f := f_1$ and suppose that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq 1$ for every $\mathfrak{p} \in \mathrm{Ass}(M/fM)$. Then $\mathrm{Ass}(M) \cap \mathrm{Ass}(M/fM) = \emptyset$. We shall show that f is a non-zero divisor for M. Suppose on the contrary that f is a zero divisor on M. By localising at a minimal prime ideal in $\mathrm{Ass}(M) \cap \mathrm{V}(Rf)$, we may assume that R is a local ring, $\mathrm{depth}_R(M) = 0$ and that $\mathrm{Ass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{m}_R\}$ with $\mathfrak{p}_i \notin \mathrm{V}(Rf)$ for all $i = 1, \ldots, m$. Then $m \geq 1$. Let $Q_1, \ldots, Q_m$ and Q be the primary components corresponding to $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ and $\mathfrak{m}_R$ respectively and let $0 = Q_1 \cap \cdots \cap Q_m \cap Q$ be an irredundant primary decomposition of the zero module in M. Let $N := Q_1 \cap \cdots \cap Q_m$. Then $N \neq 0, \mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ and f is a non-zero divisor for M/N, since $f \notin \mathfrak{p}_i$ for all $i = 1, \ldots, m$. This implies

that the canonical homomorphism $N/fN \longrightarrow M/fM$ is injective. Further, since Q is $\mathfrak{m}_R$-primary in M, we have $\mathfrak{m}_R^n N \subseteq N \cap \mathfrak{m}_R^n M \subseteq N \cap Q = 0$ for some $n \in \mathbb{N}^+$, and hence N has finite length. Therefore N/fN has finite length. But $\mathrm{depth}_R(M/fM) \geq 1$, since $\mathfrak{m}_R \notin \mathrm{Ass}(M/fM)$ and therefore cannot contain any submodules of finite length. This proves that $N/fN = 0$ and then $N = 0$ by Nakayama's lemma, which contradicts $N \neq 0$.

Proof of $(*)_r \Rightarrow (*)_{r+1}$. We may assume that R is local, $f_1, \ldots, f_{r+1} \in \mathfrak{m}_R$ and $M \neq 0$. Now, we shall prove this implication by induction on dim (R). Clearly the induction starts at dim $(R) = 0$. Put $\overline{M}_r := M/(f_1, \ldots, f_r)M$ and $\overline{M}_{r+1} := M/(f_1, \ldots, f_{r+1})M$. Then by induction hypothesis.

(†) $f_1, \ldots, f_{r+1}$ is a regular sequence on $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Supp}(\overline{M}_{r+1}) \backslash \{\mathfrak{m}_R\}$.

In particular, we have:

(††) $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq r+1$ for every $\mathfrak{p} \in \mathrm{Supp}(\overline{M}_{r+1}) \backslash \{\mathfrak{m}_R\}$.

We consider two cases:

Case 1. $\mathfrak{m}_R \in \mathrm{Ass}(\overline{M}_{r+1})$. In this case, by assumption in $(*)_{r+1}$, $\mathrm{depth}_R(M) \geq r+1$. Now, use (ii) $\Rightarrow$ (i) of the theorem to conclude that $f_1, \ldots, f_{r+1}$ is a regular sequence on M.

Case 2. $\mathfrak{m}_R \notin \mathrm{Ass}(\overline{M}_{r+1})$. In this case $\mathrm{Ass}(\overline{M}_r) \cap \mathrm{Ass}(\overline{M}_{r+1}) = \emptyset$, since $\mathrm{depth}_{R_{\mathfrak{p}}}(\overline{M}_r)_{\mathfrak{p}} \geq 1$ for every $\mathfrak{p} \in \mathrm{Ass}(\overline{M}_{r+1}) \backslash \{\mathfrak{m}_R\}$ by (††). Therefore by $(*)_1$, f_{r+1} is a non-zero divisor on $\overline{M}_r$. Now, it remains to show that the sequence $f_1, \ldots, f_r$ is a regular sequence on M. For this, let $\mathfrak{p} \in \mathrm{Ass}(\overline{M}_r)$. Since f_{r+1} is a non-zero divisor for $\overline{M}_r$, there exists $\mathfrak{q} \in \mathrm{Ass}(\overline{M}_{r+1})$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Note that $\mathfrak{q} \neq \mathfrak{m}_R$ and that $f_1, \ldots, f_r$ is a regular sequence on $M_{\mathfrak{q}}$ by (†) and hence in particular for $M_{\mathfrak{p}}$. This proves that $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq r$ for every $\mathfrak{p} \in \mathrm{Ass}(\overline{M}_r)$ and hence $f_1, \ldots, f_r$ is a regular sequence on M by $(*)_r$.

Acknowledgements

Part of this work was done while the first author was visiting Germany during April–June 2001 under a grant from DAAD, Germany. The first author thanks DAAD, Germany for financial support. The authors sincerely thank Harmut Wiebe for stimulating discussions.

References

[1] Bruns W and Herzog J, Cohen–Macaulay rings (Cambridge Studies in Advanced Mathematics 39, Cambridge: Cambridge University Press) (1993)

[2] Eisenbud D, Herrmann M and Vogel W, Remarks on regular sequences, *Nagoya Math. J.* **67** (1977) 177–180

[3] Matsumura H, Commutative ring theory (Cambridge: Cambridge University Press) (1986)

[4] Scheja G and Storch U, Regular sequences and resultants, *Res. Notes in Math.* (Natick, Massachusetts: A K Peters) (2001) vol. 8

[5] Serre J-P, Local algebra, in: Springer monographs in mathematics (Berlin, Heidelberg, New York: Springer-Verlag) (2000)